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Non-propagating solitons of the non-isospectral and variable coefficient modified KdV equation

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Abstract. The AKNS system associated with a non-isospectral and variable coefficient mKdV equation is presented. The method of inverse scattering is adapted to the non-isospectral situation to determine the time evolution of the scattering data. N-soliton solutions are obtained. Examples of oscillating or standing one-solitons with unusual dynamics are given. An in-depth study of the two-soliton case is carried out by appropriately decomposing the solution into individual soliton elements in order to examine their interactions. Breathers are also constructed.

1. Introduction

It has been observed that when a more realistic model is used to describe the propagation of waves in inhomogeneous media the equation may become explicitly dependent on time. In general the Lax pairs associated with such equations have a spectral parameter varying with time. Earlier Chen and Zheng studied the non-isospectral and variable coefficient KdV equation [1] and explicit solutions were constructed by the use of Bäcklund transformations. In [2], its initial value problem was solved by the method of inverse scattering. Solitons with non-standard dynamics were obtained, providing mathematical models for some of the non-propagating (oscillating or standing) solitons discovered experimentally [3, 4]. Indeed, if we are interested in the KdV or mKdV-type models for non-propagating solitons, constant coefficient and isospectral versions would not be sufficient since they simply do not have such solitons. It is almost necessary to consider their variable coefficient and non-isospectral extensions. However, connections of the time-dependent coefficients and non-isospectral terms to the experimental results in [3] are interesting but beyond the scope of the present paper and remain to be studied. Another viewpoint is that the additional non-isospectral terms introduced may be considered as perturbations in such a way that the resulting equation is still integrable. Furthermore, the free parametric functions can be chosen to shape some of the non-standard dynamics of the solitons, i.e. to influence their motions. For two-soliton solutions, we decomposed them into individual soliton elements and examined their interactions, leading to a better understanding of the underlying dynamics.

The aim of this article is to study the non-isospectral and varying coefficient mKdV equation by the method of inverse scattering in the spirit of [2, 5, 6]. Specifically we

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consider the following equation

$$u_t+K_0(u_{xxx}+6u^2u_x)-h(xu_x+u)+K_1u_x=0 \tag{1.1}$$

where K_0, K_1 and h are continuous functions of t. It reduces to the standard mKdV equation [7] when $K_0\equiv 1$, $K_1\equiv h\equiv 0$. Here we call (1.1) the NVmKdV equation (NV stands for non-isospectral and variable coefficients). Equation (1.1) belongs to a class of nonlinear evolution equations already studied in the literature. In particular, for equations related to the Zakharov–Shabat spectral problem, see, for example, the paper by Newell [5], and the references quoted there. Here, we explicitly allow the eigenparameter to vary according to the ordinary differential equation $\lambda_t=h(t)\lambda$. We shall show through the solutions presented in this paper that the factors that contribute to the presence of *non-propagating solitons are (i) the non-isospectral scattering problem, and (ii) the* time varying coefficients of the equation.

The paper is organized as follows. In section 1 we present the AKNS system [8] associated with (1.1). In section 2, the method of inverse scattering is adapted to the non-isospectral situation and the evolution of the scattering data is obtained. N-soliton solutions are constructed in section 3. One-soliton solutions with non-standard dynamics are presented in section 4. In section 5, an in-depth study of the two-soliton case is carried out by appropriately decomposing the solution into individual soliton elements in order to examine their interactions. It is interesting to note that the topic of structure during interaction of the soliton solutions, even for the standard KdV equation, has recently received considerable attention [9, 10]. The decomposition for the NVmKdV equation here is new and it may well be generic for most NV generalizations of other soliton equations. In section 6, solutions known as breathers are presented.

Equation (1.1) is derived as follows. We consider the following AKNS system

$$\psi_x=M\psi \qquad M=\begin{bmatrix}-\mathrm{i}\lambda & u\\ -u & \mathrm{i}\lambda\end{bmatrix} \tag{1.2}$$

$$\psi_t=N\psi \qquad N=\begin{bmatrix}A & B\\ C & -A\end{bmatrix} \tag{1.3}$$

with

$$\lambda_t=h\lambda. \tag{1.4}$$

From the consistency condition of (1.2) and (1.3)

$$M_t-N_x+MN-NM=0 \tag{1.5}$$

it follows that

$$\begin{aligned}&\mathrm{i}\lambda_t+A_x-uC-uB=0 \qquad u_t-B_x-2\mathrm{i}\lambda B-2Au=0\\ &u_t+C_x-2\mathrm{i}\lambda C+2Au=0.\end{aligned} \tag{1.6}$$

Inserting

$$\lambda_t=h\lambda \qquad A=\sum_{i=0}^{3}A_i\lambda^i \qquad B=\sum_{i=0}^{2}B_i\lambda^i \qquad \text{and } C=\sum_{i=0}^{2}C_i\lambda^i \tag{1.7}$$

into (1.6) and equating the coefficients of the powers of λ^3, λ^2, λ and λ^0, respectively, we obtain the following results:

$$\begin{aligned}&A_3 \text{ and } A_2 \text{ are independent of } x\\ &B_2=\mathrm{i}A_3u \qquad C_2=-\mathrm{i}A_3u\end{aligned}\tag{1.8}$$

$$\begin{aligned}&B_1=\mathrm{i}A_2u+A_3u_x/2 \qquad C_1=-\mathrm{i}A_2u+A_3u_x/2\\ &A_1=-\mathrm{i}hx+A_3u^2/2+\gamma_1\end{aligned}\tag{1.9}$$

where γ_1 is independent of x, and

$$\begin{aligned}&B_0=\mathrm{i}A_3u_{xx}/4+\mathrm{i}A_3u^3/2-A_2u_x/2+hxu+\mathrm{i}\gamma_1u\\ &C_0=-\mathrm{i}A_3u_{xx}/4-\mathrm{i}A_3u^3/2-A_2u_x/2-hxu-\mathrm{i}\gamma_1u\\ &A_0=-A_2u^2/2+\gamma_0\end{aligned}\tag{1.10}$$

where γ_0 is independent of x

$$u_t-\mathrm{i}A_3(u_{xxx}+6u^2u_x)/4+A_2(u_{xx}+2u^3)/2-hxu_x-hu-\mathrm{i}\gamma_1u_x-2\gamma_0u=0\tag{1.11}$$

$$u_t-\mathrm{i}A_3(u_{xxx}+6u^2u_x)/4-A_2(u_{xx}+2u^3)/2-hxu_x-hu-\mathrm{i}\gamma_1u_x+2\gamma_0u=0.\tag{1.12}$$

From (1.11) and (1.12) it follows that if $A_2=0$, $\gamma_0=0$ then (1.11) and (1.12) are the same. In particular, choosing $A_3=\mathrm{i}4K_0$ and $\gamma_1=\mathrm{i}K_1$, equation (1.11) with $A_2=0$, $\gamma_0=0$ is just the NVmKdV equation (1.1).

Then, from (1.7)–(1.10) it follows that (1.1) has the Lax pair (1.2)–(1.4) where

$$A=-\mathrm{i}4K_0\lambda^3+\mathrm{i}\lambda\,(2K_0u^2-hx+K_1)\tag{1.13}$$

$$B=4K_0u\lambda^2+2\mathrm{i}K_0u_x\lambda-K_0u_{xx}-2K_0u^3+hxu-K_1u\tag{1.14}$$

$$C=-4K_0u\lambda^2+2\mathrm{i}K_0u_x\lambda+K_0u_{xx}+2K_0u^3-hxu+K_1u.\tag{1.15}$$

2. The scattering data

Now we consider the scattering data [5, 6] of the problem (1.2) and (1.4). Suppose that

$$|xu| \qquad |u_x| \qquad \text{and } |u_{xx}|\to 0 \qquad \text{as } |x|\to\infty\tag{2.1}$$

and $f(x, y, t)$, $\bar{f}(x, y, t)$, $g(x, y, t)$ and $\bar{g}(x, y, t)$ are Jost functions of (1.2), which satisfy the following boundary conditions

$$f\to\begin{bmatrix}1\\0\end{bmatrix}\mathrm{e}^{-\mathrm{i}\lambda x} \qquad \bar{f}\to\begin{bmatrix}0\\-1\end{bmatrix}\mathrm{e}^{\mathrm{i}\lambda x} \qquad \text{as } x\to-\infty\tag{2.2}$$

$$g\to\begin{bmatrix}0\\1\end{bmatrix}\mathrm{e}^{\mathrm{i}\lambda x} \qquad \bar{g}\to\begin{bmatrix}1\\0\end{bmatrix}\mathrm{e}^{-\mathrm{i}\lambda x} \qquad \text{as } x\to\infty\tag{2.3}$$

thus

$$f=bg+a\bar{g}\to\begin{bmatrix}a\,\mathrm{e}^{-\mathrm{i}\lambda x}\\ b\,\mathrm{e}^{\mathrm{i}\lambda x}\end{bmatrix} \qquad \text{as } x\to\infty\tag{2.4}$$

where a and b depend on t and λ. From (1.13)–(1.15) it follows that as $|x|\to\infty$

$$N\to\tilde{N}=\begin{bmatrix}-4\mathrm{i}K_0\lambda^3+\mathrm{i}(-hx+K_1)\lambda & 0\\ 0 & 4\mathrm{i}K_0\lambda^3-\mathrm{i}(-hx+K_1)\lambda\end{bmatrix}. \tag{2.5}$$

Since M and N in (1.2) *and* (1.3) *satisfy the condition* (1.5), *it is easy to show that* f_t-Nf is also a solution of equation (1.2). Thus

$$f_t-Nf=\sigma_1 f+\sigma_2\bar{f} \tag{2.6}$$

where σ_1 depends on t, λ. From (1.2), (2.2) and (2.5) it follows that as $x\to-\infty$

$$\begin{aligned}f_t-Nf&\to-\mathrm{i}x\lambda_t\begin{bmatrix}1\\0\end{bmatrix}\mathrm{e}^{-\mathrm{i}\lambda x}-\tilde{N}\begin{bmatrix}1\\0\end{bmatrix}\mathrm{e}^{-\mathrm{i}\lambda x}\\ &=-\mathrm{i}hx\lambda\begin{bmatrix}1\\0\end{bmatrix}\mathrm{e}^{-\mathrm{i}\lambda x}-\begin{bmatrix}\mathrm{i}[-4K_0\lambda^3+(-hx+K_1)\lambda]\\0\end{bmatrix}\mathrm{e}^{-\mathrm{i}\lambda x}\\ &=\begin{bmatrix}-\mathrm{i}[-4K_0\lambda^3+K_1\lambda]\,\mathrm{e}^{-\mathrm{i}\lambda x}\\0\end{bmatrix}\end{aligned}$$

and

$$\sigma_1 f+\sigma_2\bar{f}\to\begin{bmatrix}\sigma_1\,\mathrm{e}^{-\mathrm{i}\lambda x}\\-\sigma_2\,\mathrm{e}^{\mathrm{i}\lambda x}\end{bmatrix}.$$

Thus, by using (2.6) we obtain

$$\sigma_2=0\qquad \sigma_1=-\mathrm{i}(-4K_0\lambda^3+K_1\lambda). \tag{2.7}$$

In view of (2.4), (2.5) and (2.7), we see that as $x\to\infty$

$$f_t-Nf\to\begin{bmatrix}(a_t-\mathrm{i}ax\lambda_t)\,\mathrm{e}^{-\mathrm{i}\lambda x}\\(b_t+\mathrm{i}bx\lambda_t)\,\mathrm{e}^{\mathrm{i}\lambda x}\end{bmatrix}-\tilde{N}\begin{bmatrix}a\,\mathrm{e}^{-\mathrm{i}\lambda x}\\b\,\mathrm{e}^{\mathrm{i}\lambda x}\end{bmatrix}=\begin{bmatrix}[a_t-\mathrm{i}a(-4K_0\lambda^3+K_1\lambda)]\,\mathrm{e}^{-\mathrm{i}x\lambda}\\ [b_t+\mathrm{i}b(-4K_0\lambda^3+K_1\lambda)]\,\mathrm{e}^{\mathrm{i}x\lambda}\end{bmatrix} \tag{2.8}$$

and

$$\sigma_1 f+\sigma_2\bar{f}=\sigma_1 f\to-\mathrm{i}(-4K_0\lambda^3+K_1\lambda)\begin{bmatrix}a\,\mathrm{e}^{-\mathrm{i}\lambda x}\\b\,\mathrm{e}^{\mathrm{i}\lambda x}\end{bmatrix}=\begin{bmatrix}-\mathrm{i}a(-4K_0\lambda^3+K_1\lambda)\,\mathrm{e}^{-\mathrm{i}\lambda x}\\-\mathrm{i}b(-4K_0\lambda^3+K_1\lambda)\,\mathrm{e}^{\mathrm{i}\lambda x}\end{bmatrix}. \tag{2.9}$$

Comparing (2.6), (2.8) and (2.9), we obtain the following time evolution equations of the scattering data a, b appearing in (2.4)

$$a_t=0 \tag{2.10}$$

i.e.

$$a=a(t,\lambda(t))=a(\lambda(0))=a\left[\lambda\exp\left(-\int_0^t h\,\mathrm{d}t\right)\right] \tag{2.11}$$

and

$$b_t=-2b\mathrm{i}(-4K_0\lambda^3+K_1\lambda) \tag{2.12}$$

i.e.

$$b=b(t,\lambda(t))=b(0,\lambda(0))\exp\left[-2\mathrm{i}\int_0^t(-4K_0\lambda^3+K_1\lambda)\,\mathrm{d}t\right]. \tag{2.13}$$

Furthermore, suppose $\lambda_j(t)$ is an eigenvalue of equation (1.2), i.e. it satisfies the following relations

$$a(\lambda_j(0))=0 \tag{2.14}$$

and

$$\lambda_j(t)=\lambda_j(0)\exp\left(\int_0^t h\,\mathrm{d}t\right) \tag{2.15}$$

by (2.11) and (1.4). Then the normalization coefficient [6]

$$\begin{aligned} c_j=c_j(t,\lambda_j(t))&=\mathrm{i}b(t,\lambda_j(t))/\dot{a}(\lambda_j(0))\exp\left(-\int_0^t h\,\mathrm{d}t\right)\\ &=c_j(0,\lambda_j(0))\exp\left[-2\mathrm{i}\int_0^t(-4K_0\lambda_j^3(t)+K_1\lambda_j(t))\,\mathrm{d}t+\int_0^t h\,\mathrm{d}t\right] \end{aligned} \tag{2.16}$$

where

$$\begin{aligned} &c_j(0,\lambda_j(0))=\mathrm{i}b(0,\lambda_j(0))/\dot{a}(\lambda_j(0))\\ &\dot{a}(\zeta)=(\mathrm{d}/\mathrm{d}\zeta)a(\zeta). \end{aligned} \tag{2.17}$$

Here (2.11), (2.13), (2.15) and (2.16) give the time evolution of the scattering data a, b and λ_j, c_j.

3. *N*-soliton solution

Now we use the scattering data c_j and $\lambda_j(t)$ to restore $u(x,t)$ in (1.1). To this end, following the inverse scattering method for the AKNS system [5, 6, 8], we obtain

$$u(x,t)=-2\mathscr{K}_1(x,x,t) \tag{3.1}$$

satisfying the initial condition

$$u(x,0)=u_0(x) \tag{3.2}$$

where $u_0(x)$ satisfies the condition (2.1) and with the initial scattering data $a(\lambda(0))$, $b(0,\lambda(0))$ and $c_j(0,\lambda_j(0))$, $\lambda_j(0)$; $\mathscr{K}_1(x,y,t)$ satisfies the Gelfand–Levitan system

$$\mathscr{K}_1(x,y,t)-F(x+y,t)-\int_x^\infty \mathscr{K}_2(x,s,t)F(s+y,t)\,\mathrm{d}s=0$$

and

$$\mathscr{K}_2(x,y,t)+\int_x^\infty \mathscr{K}_1(x,s,t)F(s+y,t)\,\mathrm{d}s=0 \qquad (x\leqslant y) \tag{3.3}$$

with

$$F(x,t)=-\sum_{j=1}^{N} c_j \exp[\mathrm{i}\lambda_j(t)x]+(\tfrac{1}{2}\pi)\int_{-\infty}^{\infty}\frac{b(t,\lambda)}{a(t,\lambda)}\mathrm{e}^{\mathrm{i}x\lambda}\,\mathrm{d}\lambda \tag{3.4}$$

and

$$\lambda_j(t)=\eta_j(t)+\mathrm{i}\xi_j(t) \qquad \xi_j(t)>0 \qquad j=1,2,\ldots,N. \tag{3.5}$$

The eigenvalues are simple and they are either pure imaginary or

$$\lambda_q(t)=-\lambda_j^*(t) \tag{3.6}$$

is also an eigenvalue (z^* is conjugate to z.), and

$$c_j=\begin{cases} c_j^* & \text{if } \lambda_j(t) \text{ is pure imaginary} \\ c_q^* & \text{if } \lambda_q(t)=-\lambda_j^*(t). \end{cases} \tag{3.7}$$

Assume that

$$b(t,\lambda)\equiv 0 \tag{3.8}$$

where λ is real, and all $\lambda_j(t)$ are pure imaginary, then $u(x,t)$ defined by (3.1) is the N-soliton solution of the NVmKdV equation (1.1).

To solve system (3.3) for $\mathscr{K}_1(x,y,t)$, we let

$$\mathscr{K}_1(x,y,t)=\sum_{j=1}^{N} k_{1j}(x,t)\exp[\mathrm{i}\lambda_j(t)y]$$

and

$$\mathscr{K}_2(x,y,t)=\sum_{j=1}^{N} k_{2j}(x,t)\exp[\mathrm{i}\lambda_j(t)y] \tag{3.9}$$

and insert them in system (3.3), to obtain the algebraic system

$$k_{1j}(x,t)+c_j\exp[\mathrm{i}x\lambda_j(t)] \\ -\sum_{s=1}^{N} k_{2s}(x,t)c_j\exp\{\mathrm{i}x[\lambda_s(t)+\lambda_j(t)]\}/\mathrm{i}[\lambda_s(t)+\lambda_j(t)]=0$$

and

$$k_{2j}(x,t)+\sum_{s=1}^{N} k_{1s}(x,t)c_j\exp\{\mathrm{i}x[\lambda_j(t)+\lambda_s(t)]\}/\mathrm{i}[\lambda_j(t)+\lambda_s(t)]=0 \\ j=1,2,\ldots,N. \tag{3.10}$$

To solve the system (3.9) for $k_{1j}(x,t)$, let

$$b_{js}=c_j\exp\{\mathrm{i}x[\lambda_j(t)+\lambda_s(t)]\}/\mathrm{i}[\lambda_j(t)+\lambda_s(t)] \tag{3.11}$$

$$\boldsymbol{B}=[b_{js}]_{N\times N} \qquad (N\times N \text{ matrix}) \tag{3.12}$$

$$\boldsymbol{A}=\begin{bmatrix} \boldsymbol{I}_N & -\boldsymbol{B} \\ \boldsymbol{B} & \boldsymbol{I}_N \end{bmatrix} \tag{3.13}$$

where I_N is the unit square matrix of order N;

$$\begin{array}{cc} & \begin{array}{cccccccccc} & & & & j\text{th column} & & & & & \end{array} \\ A_j= & \left[\begin{array}{cccccccccc} 1 & 0 & \cdots & 0 & -c_1\exp(\mathrm{i}x\lambda_1(t)) & 0 & \cdots & 0 & \vdots & \cdots \\ 0 & 1 & \cdots & 0 & -c_2\exp(\mathrm{i}x\lambda_2(t)) & 0 & \cdots & 0 & \vdots & \\ & & \cdots & & \cdots & & \cdots & & \vdots & -\boldsymbol{B} \\ 0 & & \cdots & 0 & -c_n\exp(\mathrm{i}x\lambda_N(t)) & 0 & \cdots & 1 & \vdots & \\ b_{11} & & \cdots & b_{1,j-1} & 0 & b_{1,j+1} & \cdots & b_{1N} & \vdots & \cdots \\ & & \cdots & & \vdots & & \cdots & & \vdots & \\ b_{N1} & & \cdots & b_{N,j-1} & 0 & b_{N,j+1} & \cdots & b_{NN} & \vdots & \boldsymbol{I}_N \end{array}\right] \end{array} \tag{3.14}$$

we obtain

$$k_{1j}(x,t)=\det A_j/\det A,$$

$$\mathscr{K}_1(x,y,t)=\sum_{j=1}^{N}\exp[\mathrm{i}y\lambda_j(t)]\det A_j/\det A.$$

From the above result and (3.1) it follows that the N-soliton solution of (1.1) is

$$u(x,t)=-2\sum_{j=1}^{N}\exp[\mathrm{i}x\lambda_j(t)]\det A_j/\det A. \tag{3.15}$$

4. One-soliton

Now we consider the case $N=1$. As stated in section 3, the eigenvalue $\lambda_1(t)$ is pure imaginary and

$$\lambda_1(t)=\mathrm{i}\xi_1(0)\exp\left[\int_0^t h\,\mathrm{d}t\right]$$

where $\xi_1(0)>0$. From (2.16) and (3.11) it follows that the matrix $\boldsymbol{B}$, det A, and det A_j defined by (3.12)–(3.14) become

$$\boldsymbol{B}=[b_{11}]=[(\operatorname{sgn} c_1)\exp(\phi_1)]$$

where

$$\operatorname{sgn} z=\begin{cases}-1 & \text{if } z<0\\ 0 & \text{if } z=0\\ 1 & \text{if } z>0\end{cases} \tag{4.1}$$

$$\begin{aligned}\phi_1={}&-2\xi_1(0)x\exp\left[\int_0^t h\,\mathrm{d}t\right]+2\int_0^t\left\{K_1\xi_1(0)\exp\left[\int_0^s h\,\mathrm{d}t\right]\right.\\ &\left.+4K_0\xi_1^3(0)\exp\left[3\int_0^s h\,\mathrm{d}t\right]\right\}\mathrm{d}s+\psi_1\end{aligned} \tag{4.2}$$

with

$$\psi_1=\ln|c_1(0,\mathrm{i}\xi_1(0))/2\xi_1(0)| \tag{4.3}$$

(since c_1 is real, see (3.7) in section 3) and

$$\det A=1+b_{11}^2=1+\exp(2\phi_1) \tag{4.4}$$

and

$$\begin{aligned}\det A_1&=-c_1\exp[\mathrm{i}x\lambda_1(t)]=-2\xi_1(0)\,\mathrm{sgn}[c_1(0,\mathrm{i}\xi_1(0))]\exp\left[\int_0^t h\,\mathrm{d}t\right]\\&\quad\times\exp\left\{\phi_1+\xi_1(0)x\exp\left[\int_0^t h\,\mathrm{d}t\right]\right\}.\end{aligned} \tag{4.5}$$

Thus, from (4.4), (4.5) and (3.15), we obtain the one-soliton solution of (1.1)

$$\begin{aligned}u(x,t)&=-2\exp[\mathrm{i}x\lambda_1(t)]\det A_1/\det A\\&=2\xi_1(0)\,\mathrm{sgn}[c_1(0,\mathrm{i}\xi_1(0))]\exp\left[\int_0^t h\,\mathrm{d}t\right]\mathrm{sech}(\phi_1)\end{aligned} \tag{4.6}$$

where ϕ_1 is defined by (4.2). This result is similar to (6.131) in [5].

Let us present some examples.

Example 1. Suppose $K_0=1, h=K_1=0$, then equation (1.1) reduces to the mKdV equation and (4.6) becomes

$$u(x,t)=2\xi_1(0)\,\mathrm{sgn}[c_1(0,\mathrm{i}\xi_1(0))]\,\mathrm{sech}[-2\xi_1(0)x+8\xi_1^3(0)t+\psi_1]$$

which is the one-soliton of the mKdV equation. This appeared in [7].

Example 2. Suppose $K_0=3/[4(1+t^2)]$, and $c_1(0,\mathrm{i}\xi_1(0))=2\xi_1(0)=6, h=K_1=0$, then (4.6) becomes

$$u(x,t)=6\,\mathrm{sech}(-6x+6\tan^{-1}t)$$

which is an asymptotically standing soliton as shown in figure 1.

Example 3. Suppose $K_0=3\cos t/4$, and $c_1(0,\mathrm{i}\xi_1(0))=2\xi_1(0)=6, h=K_1=0$, then (4.6) becomes

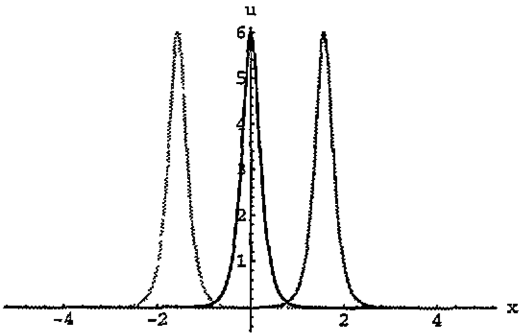


Figure 1. Asymptotically standing soliton $u(x, t)$.

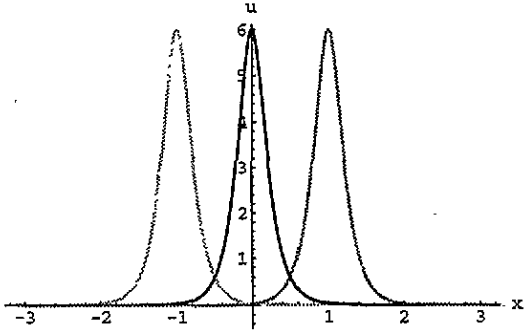


Figure 2. Oscillating soliton $u(x, t)$.

$$u(x, t) = 6\,\mathrm{sech}(-6x + 6\sin t)$$

which is an oscillating soliton as shown in figure 2. This corresponds to the result in [2].

Example 4. Suppose $K_1 = -36K_0$ and $h = 0$, $c_1(0, \mathrm{i}\xi_1(0)) = 2\xi_1(0) = 6$, then (4.6) becomes

$$u(x, t) = 6\,\mathrm{sech}(-6x)$$

which is a standing soliton. In general, from (4.2) we know that if $h = 0$ and K_0, K_1 and $\xi_1(0)$ satisfy the following condition

$$K_1 = -4K_0\xi_1^2(0) \tag{4.7}$$

then equation (1.1) has a standing soliton. Examples 1–4 are isospectral.

Next, let us consider the non-isospectral case. By direct computing, we can conclude that if K_0, K_1, h and the scattering data $\xi_1(0)$, $c_1(0, \mathrm{i}\xi_1(0))$ of $u_0(x)$ satisfy the following conditions:

$$c_1(0, \mathrm{i}\xi_1(0)) \neq 2\xi_1(0) \tag{4.8}$$

and

$$K_0 = (Ph - K_1)/4\xi_1^2(t) \tag{4.9}$$

where

$$P = \ln|c_1(0, \mathrm{i}\xi_1(0))/2\xi_1(0)|/2\xi_1(0) \tag{4.10}$$

then (1.1) has a standing one-soliton solution $u(x, t)$ defined by (4.6) with

$$\phi_1 = -2\xi_1(0)(x - P)\exp\left(\int_0^t h\,\mathrm{d}t\right). \tag{4.11}$$

Example 5. Suppose $h = -\cos t/(2 + \sin t)$

$$K_0 = (h/36)\exp\left[-2\int_0^t h\,\mathrm{d}t\right]$$

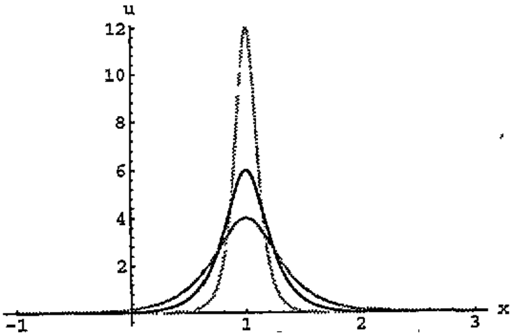


Figure 3. Standing soliton with periodically varying amplitude.

and $K_1 = 0$, $\xi_1(0) = 3$, $c_1(0, \mathrm{i}\xi_1(0)) = 6\,\mathrm{e}^6$, then (4.8) and (4.9) hold, thus, there is a standing one-soliton

$$u(x, t) = [12/(2+\sin t)]\ \mathrm{sech}\{-[12/(2+\sin t)](x-1)\}$$

with its amplitude varying periodically, as shown in figure 3.

Example 6. Suppose $h = -2t$, $K_0 = \exp(3t^2)/36$, $K_1 = 0$ and $\xi_1(0) = 3c_1(0, \mathrm{i}\xi_1(0)) = 6$, then (4.8) and (4.9) fail, thus, (4.6) becomes

$$u(x, t) = 6\exp(-t^2)\ \mathrm{sech}\{-6x\exp(-t^2) + 6t\}.$$

As t goes from $-\infty$ to $+\infty$, the wave $u(x, t)$ propagates from left to right along the x-axis and its amplitude first increases from 0 (as t goes from $-\infty$ to 0) and then decays (as t goes from 0 to $+\infty$), as shown in figure 4.

Example 7. Suppose $h = -\cos t/(2+\sin t)$, $K_0 = \cos t[(2+\sin t)/2]^3/18$ and $K_1 = 0$, $\xi_1(0) = 3$, $c_1(0, \mathrm{i}\xi_1(0)) = 6$, then (4.8) and (4.9) fail, thus (4.6) becomes

$$u(x, t) = [12/(2+\sin t)]\ \mathrm{sech}\{-[12/(2+\sin t)]x + 12\sin t\}.$$

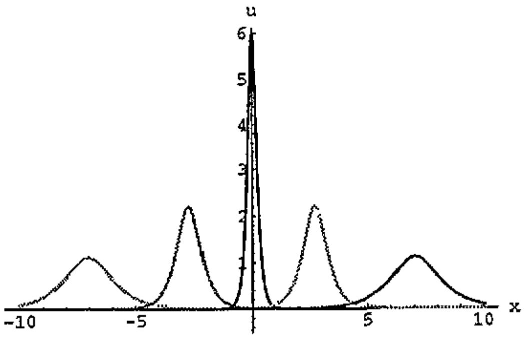


Figure 4. Soliton with decaying amplitude in both directions.

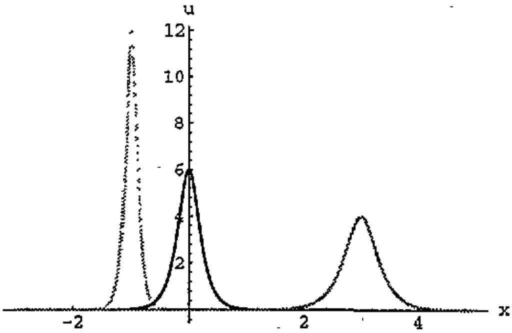


Figure 5. Periodically oscillating soliton with periodically varying amplitude.

As t goes from $-\infty$ to $+\infty$, the wave $u(x, t)$ oscillates periodically in the interval $[-1, 3]$ on the x-axis and its amplitude oscillates periodically in the interval $[4, 12]$, as shown in figure 5.

Example 8. Suppose $h=-1/(1+t^2)$, $K_0=\exp(3\tan^{-1} t)$, $K_1=0$ and $\xi_1(0)=3$, $c_1(0, \lambda_1(0))=6$, then (4.8) and (4.9) fail, thus (4.6) becomes

$$u(x, t)=6\exp[-\tan^{-1} t]\,\mathrm{sech}\{-6x\exp[-\tan^{-1} t]+6\tan^{-1} t\}.$$

As t goes from $-\infty$ to $+\infty$, the wave $u(x, t)$ is asymptotically standing and its amplitude tends to $6\exp(\pi/2)$ (or $6\exp(-\pi/2)$) as $t\to-\infty$ (or $+\infty$), as shown in figure 6.

5. Two-soliton and its decomposition

For the two-soliton solution $u(x, t)$ of (1.1), it can be seen that it separates into two single-soliton elements asymptotically for large t. Here, we are interested in a finite time decomposition of u into two individual soliton elements and they coincide with the two soliton elements for large t. This decomposition shows clearly the structure of the two-soliton solution during interaction.

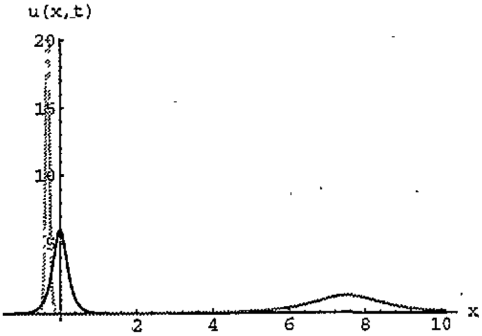


Figure 6. Asymptotically standing soliton with asymptotically varying amplitude in both directions.

For the case $N=2$, then from (2.16) and (3.11) the det A of matrix A defined by (3.13) becomes

$$\begin{aligned}\det A &= 1 - c_1^2 \exp[\mathrm{i}4\lambda_1 x]/4\lambda_1^2 - c_2^2 \exp[\mathrm{i}4\lambda_2 x]/4\lambda_2^2 \\ &\quad - \{1 - [(\lambda_1-\lambda_2)/(\lambda_1+\lambda_2)]^2\} c_1 c_2 \exp[\mathrm{i}2(\lambda_1+\lambda_2)x]/2\lambda_1\lambda_2 \\ &\quad + [(\lambda_1-\lambda_2)/(\lambda_1+\lambda_2)]^4 c_1^2 c_2^2 \exp[\mathrm{i}4(\lambda_1+\lambda_2)x]/16\lambda_1^2\lambda_2^2 \\ &= [1+\exp(2\phi_1)][1+\exp(2\phi_2)] + [(1-\mathscr{A})/\mathscr{A}][\exp(\phi_1)+\exp(\phi_2)]^2 \qquad (5.1)\end{aligned}$$

where $\lambda_j=\lambda_j(t)$, $j=1, 2, \ldots$, and

$$\mathscr{A}^{1/2} = [(\lambda_1-\lambda_2)/(\lambda_1+\lambda_2)] = \{[\lambda_1(0)-\lambda_2(0)]/[(\lambda_1(0)+\lambda_2(0)]\} = \exp(\gamma) \qquad (5.2)$$

$$c_j(0, \lambda_j(0))/2\mathrm{i}\lambda_j(0) = \exp(\psi_j) \qquad (5.3)$$

$$\phi_j = \mathrm{i}2\lambda_j x - 2\mathrm{i}\int_0^t (-4K_0\lambda_j^3 + K_1\lambda_j)\,\mathrm{d}t + \psi_j + \gamma. \qquad (5.4)$$

Similarly, we obtain

$$\begin{aligned}&\exp(\mathrm{i}\lambda_1 x)\det A_1 \\ &\quad = -[\mathrm{i}2\lambda_1/\mathscr{A}^{(1/2)}]\exp(\phi_1)\{1+[(1-\mathscr{A})/\mathscr{A}]\exp(\phi_1+\phi_2)+\exp(2\phi_2)/\mathscr{A}\} \\ &\qquad + \mathrm{i}(\lambda_1+\lambda_2)[(1-\mathscr{A})/\mathscr{A}]\exp(\phi_1+\phi_2)[\exp(\phi_1)+\exp(\phi_2)]/\mathscr{A}^{1/2} \\ &\exp(\mathrm{i}\lambda_2 x)\det A_2 \\ &\quad = -[\mathrm{i}2\lambda_2/\mathscr{A}^{(1/2)}]\exp(\phi_2)\{1+[(1-\mathscr{A})/\mathscr{A}]\exp(\phi_1+\phi_2)+\exp(2\phi_2)/\mathscr{A}\} \\ &\qquad + i(\lambda_1+\lambda_2)[(1-\mathscr{A})/\mathscr{A}]\exp(\phi_1+\phi_2)[\exp(\phi_1)+\exp(\phi_2)]/\mathscr{A}^{1/2}. \qquad (5.5)\end{aligned}$$

From (5.1)–(5.5) and (3.15) it follows that

$$\begin{aligned}u(x,t) &= -2[\exp(\mathrm{i}\lambda_1 x)\det A_1 + \exp(\mathrm{i}\lambda_2 x)\det A_2]/\det A \\ &= [-2\mathrm{i}\lambda_1/\mathscr{A}^{(1/2)}]\cosh\phi_2/\Delta + [-2\mathrm{i}\lambda_2/\mathscr{A}^{(1/2)}]\cosh\phi_1/\Delta \qquad (5.6)\end{aligned}$$

where

$$\Delta = (\cosh\phi_1)(\cosh\phi_2) + [(1-\mathscr{A})/2\mathscr{A}][1+\cosh(\phi_1-\phi_2)]. \qquad (5.7)$$

Now we consider the case that the eigenvalues $\lambda_1(t)$ and $\lambda_2(t)$ i.e. $\lambda_1(0)$ and $\lambda_2(0)$, both are pure imaginary. Without loss of generality we assume

$$\lambda_1(0)=\mathrm{i}\xi_{10} \qquad \lambda_2(0)=\mathrm{i}\xi_{20} \qquad \xi_{10}>\xi_{20}>0. \qquad (5.8)$$

Then, it is easy to show that γ, ψ_j and ϕ_j defined by (5.2)–(5.4) become

$$\mathscr{A}^{1/2} = [(\lambda_1-\lambda_2)/(\lambda_1+\lambda_2)] = [(\xi_{10}-\xi_{20})/(\xi_{10}+\xi_{20})] = \exp(\gamma) \qquad (5.9)$$

$$|c_j(0,\mathrm{i}\xi_{j0})|/2\xi_{j0} = \exp(\psi_j) \qquad (5.10)$$

$$\begin{aligned}\phi_j &= -2\xi_{j0}x\exp\left(\int_0^t h\,\mathrm{d}t\right) + 2\int_0^t\left[4K_0\xi_{j0}^3\exp\left(3\int_0^s h\,\mathrm{d}t\right) + K_1\xi_{j0}\exp\left(\int_0^s h\,\mathrm{d}t\right)\right]\mathrm{d}s \\ &\quad + \psi_j + \gamma + \mathrm{i}\{1-\operatorname{sgn}[c_j(0,\mathrm{i}\xi_{j0})]\}\pi/2. \qquad (5.11)\end{aligned}$$

Thus, $u(x, t)$ is real and is a two-soliton. (5.6) may be rewritten as below

$$u(x, t)=u_1(x, t)+u_2(x, t) \tag{5.12}$$

where

$$u_1(x, t)=\left[2\xi_{10}\exp\left(\int_0^t h\,\mathrm{d}t\right)\Big/\mathscr{A}^{(1/2)}\right]\rho(\phi_1, \phi_2)\operatorname{sech}(\phi_1)$$

and

$$u_2(x, t)=\left[2\xi_{20}\exp\left(\int_0^t h\,\mathrm{d}t\right)\Big/\mathscr{A}^{(1/2)}\right]\rho(\phi_1, \phi_2)\operatorname{sech}(\phi_2) \tag{5.13}$$

ϕ_j is defined by (5.11); sgn z is defined by (4.1);

$$\rho(\phi_1, \phi_2)=1/\{1+[(1-\mathscr{A})/2\mathscr{A}][1+\cosh(\phi_1-\phi_2)]\operatorname{sech}(\phi_1)\operatorname{sech}(\phi_2)\}. \tag{5.14}$$

In particular, if $h=K_1=0$ and $K_0=1$, then (5.12) reduces to the two-soliton of mKdV equation [6].

Formula (5.12) with (5.13) and (5.14) decomposes the two-soliton $u(x, t)$ of equation (1.1) into individual solitary waves $u_1(x, t)$ and $u_2(x, t)$. In the spirit of [2], such decomposition is used to determine the time t_d and the coordinate x_d at which the solitary waves $u_1(x, t)$ and $u_2(x, t)$ interact, during which the two-soliton becomes a single peak solitary wave. t_d and x_d satisfy the system

$$\phi_j=0 \qquad j=1, 2 \tag{5.15}$$

where ϕ_j is defined by (5.11). We illustrate these by some examples.

Example 9. Suppose $h=0$ (isospectral), $K_1=-36K_0=-9\cos t/2$ and $\xi_{10}=3$, $\xi_{20}=1$, $c_1(0, \mathrm{i}\xi_{10})=12$, $c_2(0, \mathrm{i}\xi_{20})=4$. Then the system (5.15) becomes

$$\begin{cases}\phi_1=-6x-\ln 2=0\\ \phi_2=-2x-8\sin t-\ln 2=0.\end{cases}$$

Solving this, we obtain $t_d\simeq-0.058+2p\pi$ or $t_d\simeq0.058+(2p-1)\pi$, $p=0$, $\pm1, \pm2, \dots, \pm n, \dots$, and $x_d\simeq-0.1155$. Then

$$u_1(x, t)=12\rho(\phi_1, \phi_2)\operatorname{sech}(-6x-\ln 2)$$

$$u_2(x, t)=4\rho(\phi_1, \phi_2)\operatorname{sech}(-2x-\ln 2-8\sin t)$$

where

$$\rho(\phi_1, \phi_2)=1/\{1+1.5[1+\cosh(-4x+8\sin t)]$$
$$\times\operatorname{sech}(-6x-\ln 2)\operatorname{sech}(-2x-\ln 2-8\sin t)\}.$$

Figure 7 is the graph of the two-soliton $u(x, t)$ travelling along the x-axis. It shows that in the time period $[-\pi/2, 3\pi/2]$, u_1 stays fixed at $x=x_d=-\ln 2/6$ while u_2 oscillates about u_1 between $x_{\min}\simeq-7$ and $x_{\max}\simeq6$ with period 2π, where u_1 is the higher wave

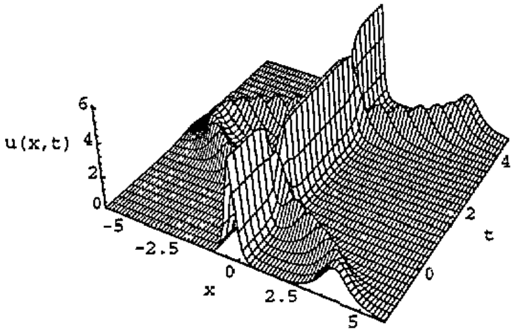


Figure 7. Periodically oscillating two-soliton $u=u_1+u_2$.

and u_2 is the lower wave as shown in figure 7, and u_1 and u_2 interact twice each period, since t_d has two values in a period.

Example 10. Suppose $h=0$ (isospectral), $K_1=-36K_0=9t\exp(-t^2)$ and $\xi_{10}=3$, $\xi_{20}=1$, $c_1(0, \mathrm{i}\xi_{10})=6$, $c_2(0, \mathrm{i}\xi_{20})=2$. Then the system (5.15) becomes

$$\begin{cases}\phi_1=-6x-\ln 2=0\\ \phi_2=-2x-\ln 2-8[\exp(-t^2)-1]=0.\end{cases}$$

Solving this, we obtain $t_d\simeq\pm 0.24$ and $x_d\simeq -0.1155$. Then

$$u_1(x, t)=12\rho(\phi_1, \phi_2)\,\mathrm{sech}(-6x-\ln 2),$$

$$u_2(x, t)=4\rho(\phi_1, \phi_2)\,\mathrm{sech}\{-2x-\ln 2-8[\exp(-t^2)-1]\}$$

where

$$\rho(\phi_1, \phi_2)=1/\{1+1.5[1+\cosh(-4x+8(\exp(-t^2)-1))]\\ \times\mathrm{sech}(-6x-\ln 2)\,\mathrm{sech}(-2x-\ln 2-8(\exp(-t^2)-1))\}.$$

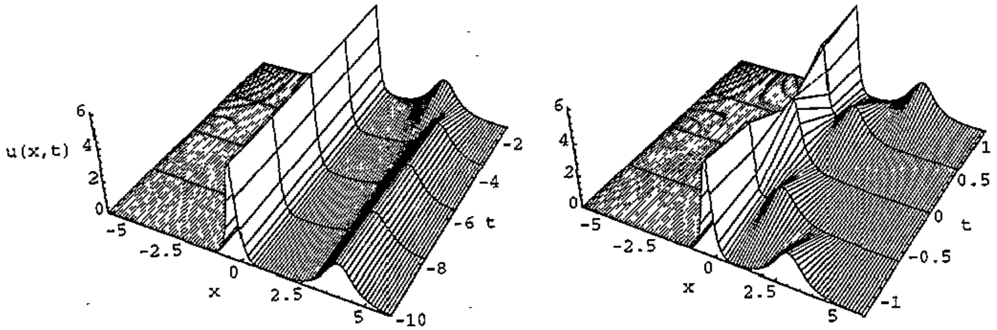
Figure 8(a)–(c) shows that for $t\leqslant -10$, the two-soliton $u(x, t)$ is in the finite limiting position—interval $[-0.35, 7.5]$ (as $t\to -\infty$) on the x-axis, the left wave is just u_1 and the right wave is just u_2. For $-10<t<t_d(\simeq -0.24)$, u and u_2 travel to the left but u_1 stands still until u_1 and u_2 interact. For t near -0.24, at the point x near $x_d\simeq -0.1155$, u becomes a single-peak wave whose amplitude decreases because u_1 and u_2 interact here, as shown in figure 8(b). (In this case, the amplitude of u_1 also decreases and u_2 changes from a single-peak wave to a double-peak wave, not shown in figure 8(b).) After $t=0.24$, i.e. for $0.24<t<+\infty$, u travels back to the original position (as $t\to -\infty$) on the x-axis and so does u_2 and u_1 always stands at the original position, as shown in figure 8(c). That is, u is an asymptotically standing wave.

Similar to section 4, it is easy to show that if K_0, K_1, h and the scattering data $\xi_{j0}=\xi_j(0)$, $c_j(0, \mathrm{i}\xi_{j0})$ of $u_0(x)$, $j=1, 2, \ldots$, satisfy the following conditions:

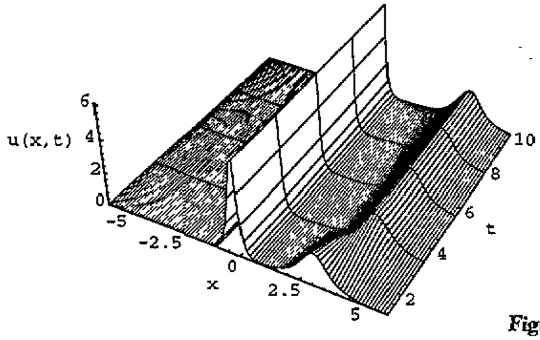
$$|c_j(0, \mathrm{i}\xi_{j0})|/2\xi_{j0}=\mathrm{e}^{-\gamma} \tag{5.16}$$

and

$$K_0=(P_jh-K_1)/4\xi_j^2(t) \tag{5.17}$$



(*a*) $t=-10, -7.8, -5.6, -3.4, -1.2$ (*b*) $t=-1.2, -0.6, 0, 0.6, 1.2$



(*c*) $t=1.2, 3.4, 5.6, 7.8, 10$

Figure 8. Asymptotically standing two-soliton $u=u_1+u_2$.

where γ is as in (5.9) and

$$P_j=[\ln|c(0, \mathrm{i}\xi_{j0})/2\xi_{j0}|+\gamma]/2\xi_{j0} \tag{5.18}$$

then (1.1) has a standing two-soliton solution $u(x, t)$ defined by (5.12) and (5.13) with

$$\phi_j=-2\xi_{j0}(x-P_j)\exp\left(\int_0^t h\,\mathrm{d}t\right). \tag{5.19}$$

Example 11. (standing two-soliton). Suppose $h=-\cos t/(2+\sin t)$ (non-isospectral), $K_1=0$, and

$$K_0=h\exp\left(-2\int_0^t h\,\mathrm{d}t\right)\Big/12$$

$\xi_{10}=3$, $\xi_{20}=1$, $c_1(0, \mathrm{i}\xi_{10})=12\,\mathrm{e}^{18}$, $c_2(0, \mathrm{i}\xi_{20})=4\,\mathrm{e}^{2/3}$. Then (5.16)–(5.18) hold, thus, the system (5.15) becomes (5.19), i.e.

$$\begin{cases}\phi_1=-12(x-3)/(2+\sin t)=0\\ \phi_2=-4(x-1/3)/(2+\sin t)=0.\end{cases}$$

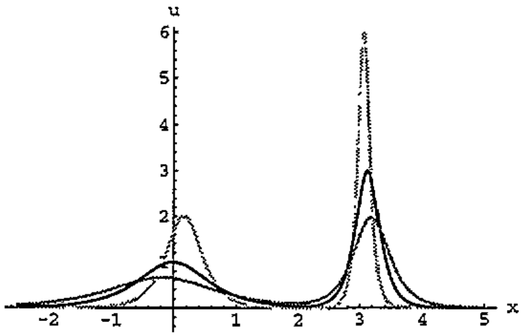


Figure 9. Standing two-soliton $u = u_1 + u_2$.

This system is not compatible, i.e. the waves $u_1(x, t)$ and $u_2(x, t)$ do not interact at all and are standing solitary waves. Here

$$u_1(x, t) = [12/(2+\sin t)]\rho(\phi_1, \phi_2)\,\mathrm{sech}[-12(x-3)/(2+\sin t)],$$

$$u_2(x, t) = [4/(2+\sin t)]\rho(\phi_1, \phi_2)\,\mathrm{sech}[-4(x-1/3)/(2+\sin t)]$$

where

$$\rho(\phi_1, \phi_2) = 1/\{1 + 1.5\{1 + \cosh[-8(x-13/3)/(2+\sin t)]\} \times \mathrm{sech}[-12(x-3)/(2+\sin t)]\,\mathrm{sech}[-4(x-1/3)/(2+\sin t)]\}.$$

Figure 9 shows that for all times, the two-soliton $u(x, t)$ stands in the interval $[-2, 5]$ on the x-axis while $u_1(x, t)$ (the right wave) stands in $[2, 5]$ and $u_2(x, t)$ (the left wave) stands in $[-2, 2]$. But their amplitudes oscillate periodically.

Periodically oscillating or asymptotically standing two-solitons which are similar to that of [2] also exist for NVmKdV equation (1.1). But we omit them here.

We have demonstrated in the above examples that the dynamics of the solitons of the NVmKdV equation is richer than that of their counterparts for the standard mKdV equation. Thus, their motion can be oscillatory, standing, asymptotically standing, amplitude pulsating and more. This is also true for the KdV case. The reader can easily convince themselves of this by adjusting appropriately the coefficients and the non-isospectral terms even though not all of them were shown in the previous paper [2]. In the following section we present breather-type solutions which are not present in the KdV case.

6. Further result (breather solution)

Here we consider the case $N=2$ and $\lambda_2 = -\lambda_1^*$, that is

$$\lambda_j = \lambda_j(t) = \eta(-1)^j + \mathrm{i}\xi = [\eta_0(-1)^j + \mathrm{i}\xi_0]\exp\left(\int_0^t h\,\mathrm{d}t\right) \qquad (6.1)$$

where $j = 1, 2$; $\eta_0, \xi_0 > 0$; Then, $\mathscr{A}$ and γ, ψ_j defined by (5.2), (5.3) become

$$\mathscr{A}^{1/2} = [(\lambda_1 - \lambda_2)/(\lambda_1 + \lambda_2)] = \mathrm{i}(\eta_0/\xi_0) = \exp(\gamma) \qquad (6.2)$$

with

$$\gamma = \ln(\eta_0/\xi_0) + \mathrm{i}\pi/2 \tag{6.3}$$

and

$$\begin{aligned}\exp(\psi_j) &= [c_j(0, \lambda_j(0))/2\mathrm{i}\lambda_j(0)] \\ &= [c_j(0, \lambda_j(0))]/2[-\xi_0 + \mathrm{i}\eta_0(-1)^j]\end{aligned} \tag{6.4}$$

with

$$\begin{aligned}\psi_j = {}& \ln|c_1(0, \lambda_1(0))| - \ln 2 - (1/2)\ln(\xi_0^2 + \eta_0^2) \\ & -\mathrm{i}(-1)^j \operatorname{Arg}[c_1(0, \lambda_1(0))] + \mathrm{i}(-1)^j[\tan^{-1}(\eta_0/\xi_0) - \pi]\end{aligned} \tag{6.5}$$

(since $c_2(0, \lambda_2(0)) = c_1^*(0, \lambda_1(0))$, see (3.7) in section 3).

Let

$$\psi' = -\ln 2 - (1/2)\ln(\xi_0^2 + \eta_0^2) + \ln(\eta_0/\xi_0) \tag{6.6}$$

then by using (5.4) and the above results we obtain

$$\begin{aligned}\psi_j + \gamma = {}& \ln|c_1(0, \lambda_1(0))| + \psi' \\ & + \mathrm{i}\{\pi/2 + (-1)^j[-\pi + \tan^{-1}(\eta_0/\xi_0)] - \operatorname{Arg}[c_1(0, \lambda_1(0))]\}\end{aligned}$$

and

$$\begin{aligned}\phi_j = {}& 2[-\xi_0 + \mathrm{i}\eta_0(-1)^j] \times \exp\left(\int_0^t h\,\mathrm{d}t\right) - 2\mathrm{i}\int_0^t \left\{-4K_0[\eta_0(-1)^j + \mathrm{i}\xi_0]^3 \exp\left(3\int_0^s h\,\mathrm{d}t\right)\right. \\ & \left. + K_1[\eta_0(-1)^j + \mathrm{i}\xi_0]\exp\left(\int_0^s h\,\mathrm{d}t\right)\right\}\mathrm{d}s + \ln|c_1(0, \lambda_1(0))| \\ & + \psi' + \mathrm{i}\{\pi/2 + (-1)^j[-\pi + \tan^{-1}(\eta_0/\xi_0)] - \operatorname{Arg}[c_1(0, \lambda_1(0))]\} \\ = {}& \phi' + \mathrm{i}\{(-1)^j\phi'' - \pi(-1)^j + \pi/2\}\end{aligned} \tag{6.7}$$

where

$$\begin{aligned}\phi' = {}& -2\xi_0 \times \exp\left(\int_0^t h\,\mathrm{d}t\right) + 2\int_0^t \left\{4K_0[\xi_0^3 - 3\xi_0\eta_0^2]\exp\left(3\int_0^s h\,\mathrm{d}t\right)\right. \\ & \left. + K_1\xi_0 \exp\left(\int_0^s h\,\mathrm{d}t\right)\right\}\mathrm{d}s + \ln|c_1(0, \lambda_1(0))| + \psi'\end{aligned} \tag{6.8}$$

$$\begin{aligned}\phi'' = {}& 2\eta_0 x \exp\left(\int_0^t h\,\mathrm{d}t\right) + 2\int_0^t \left\{4K_0[\eta_0^3 - 3\eta_0\xi_0^2]\exp\left(3\int_0^s h\,\mathrm{d}t\right)\right. \\ & \left. - K_1\eta_0 \exp\left(\int_0^s h\,\mathrm{d}t\right)\right\}\mathrm{d}s - \operatorname{Arg}[c_1(0, \lambda_1(0))] + \tan^{-1}(\eta_0/\xi_0).\end{aligned} \tag{6.9}$$

Hence, we have

$$\begin{aligned}\cosh \phi_j &= \cosh\{\phi' + \mathrm{i}[(-1)^j \phi'' - \pi(-1)^j + \pi/2]\}\\ &= (-1)^j \cosh \phi' \sin \phi'' - \mathrm{i} \sinh \phi' \cos \phi''\end{aligned} \tag{6.10}$$

and

$$\cosh(\phi_1 - \phi_2) = \cosh[\mathrm{i}(-2\phi'' + 2\pi)] = \cos(-2\phi'' + 2\pi) = \cos 2\phi''. \tag{6.11}$$

From (5.7), (6.2), (6.10) and (6.11), it follows that

$$\begin{aligned}\Delta &= (\cosh \phi_1)(\cosh \phi_2) + [(1-\mathscr{A})/2\mathscr{A}][1 + \cosh(\phi_1 - \phi_2)]\\ &= -(\cosh^2 \phi' \sin^2 \phi'' + \sinh^2 \phi' \cos^2 \phi'') - [(\xi_0^2 + \eta_0^2)/\eta_0^2] \cos^2 \phi''\\ &= -[\cosh^2 \phi' + (\xi_0^2/\eta_0^2) \cos^2 \phi''].\end{aligned} \tag{6.12}$$

Furthermore, it is easy to show that

$$\begin{aligned}&-(2\mathrm{i}\lambda_1/A^{1/2}) \cosh \phi_2\\ &\quad = \exp\left[-\psi' + \int_0^t h \,\mathrm{d}t + \mathrm{i} \tan^{-1}(\eta_0/\xi_0) + \mathrm{i}\pi/2\right]\\ &\qquad \times [\cosh \phi' \sin \phi'' + \mathrm{i} \sinh \phi' \cos \phi'']\\ &\quad = \{[-\xi_0 \sinh \phi' \cos \phi'' - \eta_0 \cosh \phi' \sin \phi'']\\ &\qquad + \mathrm{i}[-\eta_0 \sinh \phi' \cos \phi'' + \xi_0 \cosh \phi' \sin \phi'']\}\\ &\qquad \times (\xi_0^2 + \eta_0^2)^{-1/2} \exp\left(-\psi' + \int_0^t h \,\mathrm{d}t\right)\end{aligned} \tag{6.13}$$

and

$$\begin{aligned}&-(2\mathrm{i}\lambda_2/A^{(1/2)}) \cosh \phi_1\\ &\quad = \{[-\xi_0 \sinh \phi' \cos \phi'' - \eta_0 \cosh \phi' \sin \phi'']\\ &\qquad - \mathrm{i}[-\eta_0 \sinh \phi' \cos \phi'' + \xi_0 \cosh \phi' \sin \phi'']\}\\ &\qquad \times (\xi_0^2 + \eta_0^2)^{-1/2} \exp\left(-\psi' + \int_0^t h \,\mathrm{d}t\right).\end{aligned} \tag{6.14}$$

By using (5.6), (6.6), (6.12), (6.13) and (6.14) we obtain another type of solution of the NVmKdV equation (1.1)

$$\begin{aligned}u(x, t) &= \{-(2\mathrm{i}\lambda_1/A^{1/2}) \cosh \phi_2 - (2\mathrm{i}\lambda_2/A^{(1/2)}) \cosh \phi_1\}/\Delta\\ &= 2(\xi_0^2 + \eta_0^2)^{-1/2} \exp\left(-\psi' + \int_0^t h \,\mathrm{d}t\right) \frac{\xi_0 \sinh \phi' \cos \phi'' + \eta_0 \cosh \phi' \sin \phi''}{\cosh^2 \phi' + (\xi_0^2/\eta_0^2) \cos^2 \phi''}\\ &= 4\xi_0 \exp\left(\int_0^t h \,\mathrm{d}t\right) \operatorname{sech} \phi' \frac{\sin \phi'' + (\xi_0/\eta_0) \tanh \phi' \cos \phi''}{1 + (\xi_0^2/\eta_0^2) \cos^2 \phi'' \operatorname{sech}^2 \phi'}\end{aligned} \tag{6.15}$$

which is the so-called breather solution [7].

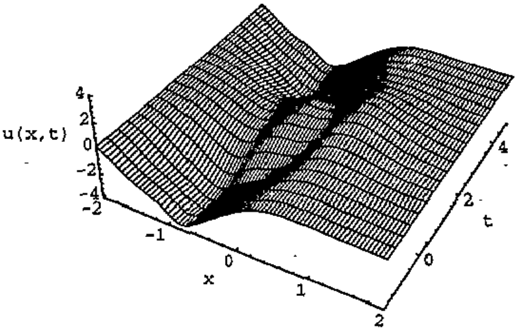


Figure 10. Periodically oscillating breather $u(x, t)$.

In particular, if $h = K_1 = 0$ and $K_0 = 1$, then (6.15) reduces to a breather solution of the mKdV equation in [7].

The wave $u(x, t)$ defined by (6.15) may also be periodically oscillating or asymptotically standing. We give the following examples.

Example 12 (periodically oscillating breather). Suppose $h = 0$, $K_1 = -8K_0 = \cos t/4$, $\xi_0 = \eta_0 = 1$ and $c_1(0, \lambda_1(0)) = \exp(i\pi/4)$. Then (6.15) becomes

$$u(x, t) = 4\,\frac{\sin 2x + \tanh(-2x + \sin t - 1.5 \ln 2) \cos 2x}{1 + \cos^2 2x\, \mathrm{sech}^2(-2x + \sin t - 1.5 \ln 2)}\, \mathrm{sech}(-2x + \sin t - 1.5 \ln 2).$$

Figure 10 shows the situation for $u(x, t)$ in a certain interval of time.

Example 13 (standing breather). Suppose $h = 0$, $K_1 = 8K_0 = \cos t/4$, $\xi_0 = \eta_0 = 1$ and $c_1(0, \lambda_1(0)) = \exp(i\pi/4)$. Then (6.15) becomes

$$u(x, t) = 4\,\frac{\sin(2x - \sin t) + \tanh(-2x - 1.5 \ln 2) \cos(2x - \sin t)}{1 + \cos^2(2x - \sin t)\, \mathrm{sech}^2(-2x - 1.5 \ln 2)}\, \mathrm{sech}(-2x - 1.5 \ln 2).$$

Figure 11 shows the situation for $u(x, t)$ in a certain interval of time.

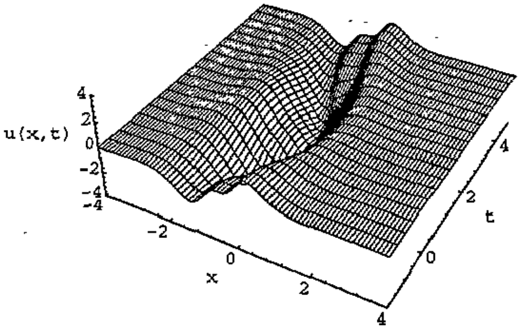


Figure 11. Standing breather $u(x, t)$.

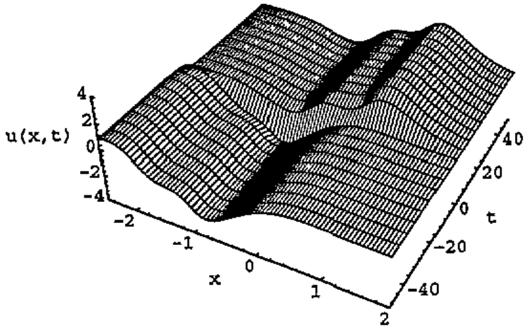


Figure 12. Asymptotically standing breather $u(x, t)$.

Example 14 (asymptotically standing breather). Suppose $h=0$, $K_1=-8K_0=1/4(1+t^2)$, $\xi_0=\eta_0=1$ and $c_1(0, \lambda_1(0))=\exp(\mathrm{i}\pi/4)$. Then (6.15) becomes

$$u(x, t)=4\,\frac{\sin 2x+\tanh(-2x+\tan^{-1} t-1.5 \ln 2) \cos 2x}{1+\cos^2 2x\, \mathrm{sech}^2(-2x+\tan^{-1} t-1.5 \ln 2)}\, \mathrm{sech}(-2x+\tan^{-1} t-1.5 \ln 2).$$

Figure 12 shows the situation for $u(x, t)$.

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